A Generalization of the Motzkin-Taussky Theorem

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ABSTRACT

In this paper we generalize the Motzkin-Taussky theorem to matrices with polynomial entries.

1. INTRODUCTION

Let A_0, A_1 be $n\times n$ complex valued matrices. Denote

$$A(\alpha,\beta)=\alpha A_0+\beta A_1. \tag{1.1}$$

A remarkable theorem of Motzkin and Taussky [4] asserts that if the pencil $A(\alpha,\beta)$ is diagonable for all $\alpha,\beta\in\mathbb{C}$ [all $A(\alpha,\beta)$ similar to a diagonal matrix], then there exists a nonsingular matrix P such that

$$P^{-1}A(\alpha,\beta)P=\alpha D_0+\beta D_1, \tag{1.2}$$

where D_0, D_1 are diagonal matrices. This in particular implies that the eigenvalues of $A(\alpha,\beta)$ are linear in α,β—i.e., $A(\alpha,\beta)$ possess the L-property. The L-property of the pencil $A(\alpha,\beta)$ follows from a milder assumption, namely, that $A(\alpha,\beta)$ is diagonable except when α/β is a given complex number on the Riemann sphere [for example, it suffices to assume that $A(\alpha,1)$ is diagonable for any $\alpha\in\mathbb{C}$]. The original proof of the Motzkin-Taussky theorem relies heavily on algebraic geometry. In [1, II, 2.5] Kato gives a different proof of these results based upon the theory of complex functions in one variable. In [2] the results of Motzkin and Taussky were used as follows:

COROLLARY 1.3. *Consider a pencil $A(\alpha,\beta)$ when A_0 and B_0 are noncommuting diagonable matrices. Then there exists a complex number α_0 such that*

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$A(\alpha_0,1)$ is not diagonable. If A_0 and A_1 are noncommuting hermitian matrices, then there exist two distinct points α_0, α_1 such that $A(\alpha_j,1)$ is not diagonable for $j=0,1$.

To deduce the second part of Corollary 1.3 one needs to combine the theorems mentioned above with an earlier result of Motzkin and Taussky [3] which claims that the L-property of $A(\alpha,\beta)$ for hermitian A_0, A_1 implies (1.2).

The numbers α_0 for which $A(\alpha_0,1)$ is not diagonable have physical significance in the theory of atomic autoionization resonances [2]. In what follows we generalize the first theorem of Motzkin and Taussky mentioned here to polynomials

$$A(z)=\sum_{k=0}^{m} z^k A_k. \tag{1.4}$$

2. THE RESULT

As usual, let $M_n(\mathbb{C})$ denote the $n\times n$ complex valued matrices. Consider the polynomial $A(z)$ (1.4) where $A_k\in M_n(\mathbb{C})$, $k=0,\dots,m$. Let

$$p(\lambda,z)=|\lambda I-A(z)|=\lambda^n+\sum_{j=1}^{n} a_j(z)\lambda^{n-j}. \tag{2.1}$$

Clearly $a_j(z)$ is a polynomial in z for $j=1,\dots,n$. Thus the eigenvalues $\lambda_1(z),\dots,\lambda_n(z)$ are algebraic functions of z which satisfy

$$p(\lambda,z)=0. \tag{2.2}$$

So for any $z_0\in\mathbb{C}$ one has the Puiseux series

$$\lambda_j(z)=\sum_{k=0}^{\infty}\lambda_{jk}(z-z_0)^{k/s}, \qquad j=1,\dots,n, \tag{2.3}$$

for some integer s which can be chosen independently of z_0. Around $z_0=\infty$ one has the expansion

$$\lambda_j(z)=z^m\sum_{k=0}^{\infty}\lambda_{jk}z^{-k/s}. \tag{2.4}$$

See for example [1, II, 1.2].

DEFINITION 2.5. Let $\lambda_p(z)$ and $\lambda_q(z)$ be two distinct eigenvalues of $A(z)$ such that $\lambda_p(z) \not\equiv \lambda_q(z)$ in the neighborhood of z_0. The eigenvalues $\lambda_p(z)$ and $\lambda_q(z)$ are said to touch at z_0 if

$$\lim_{z \to z_0} \frac{\lambda_p(z) - \lambda_q(z)}{z - z_0} = 0 \tag{2.6}$$

for a finite z_0 and

$$\lim_{z \to \infty} \frac{\lambda_p(z) - \lambda_q(z)}{z^{m-1}} = 0 \tag{2.7}$$

if $z_0 = \infty$, where A_m appearing in (1.4) is not a zero matrix.

THEOREM 2.8. *Let $A(z)$ be an $n \times n$ matrix polynomial of the form* (1.4). *Assume that $A(z)$ is a diagonable matrix for each finite z, and assume furthermore that A_m is a nonzero diagonable matrix. Then either there exists z_0 (finite or infinite) and two distinct eigenvalues $\lambda_p(z)$ and $\lambda_q(z)$ which touch at z_0, or there exists a nonsingular matrix $P \in M_n(\mathbb{C})$ such that*

$$P^{-1}A(z)P = \sum_{k=0}^{m} z^k D_k, \tag{2.9}$$

where $D_0, \ldots, D_m$ are diagonal matrices.

Proof. The matrix $A(z)$ has l distinct eigenvalues $\mu_1(z), \ldots, \mu_l(z)$, such that $\mu_k(z)$ has multiplicity n_k, $k = 1, \ldots, l$. That is, for all values of z_0 except a finite number of points $\zeta_1, \ldots, \zeta_r$ (one of these points may be infinite), $A(z_0) \in M_n(\mathbb{C})$ has exactly l distinct eigenvalues $\mu_1(z_0), \ldots, \mu_l(z_0)$. Suppose first that $z \neq \zeta_j$, $j = 1, \ldots, r$. Let $P_j(z)$ be the projection of $A(z)$ on $\mu_j(z)$. That is,

$$P_j(z) = \prod_{k=1,\, k \neq j}^{l} \frac{A(z) - \mu_k(z)I}{\mu_j(z) - \mu_k(z)}. \tag{2.10}$$

Since $\mu_k(z)$, $k = 1, \ldots, l$ are s-multivalued (at most) analytic functions in z, $P_j(z)$ is at most an s-multivalued analytic function of z for $z \neq \zeta_j$, $j = 1, \ldots, r$. Suppose that no two distinct eigenvalues of $A(z)$ touch at any point of the Riemann sphere. We claim that $P_j(z)$ is a bounded s-multivalued (at most) analytic function on the Riemann sphere. Indeed, assume for simplicity of notation that $j=1$ and for some finite ζ

$$\eta=\mu_1(\zeta)=\mu_2(\zeta)=\cdots=\mu_u(\zeta),$$
$$\mu_k(\zeta)\neq\mu_1(\zeta), \qquad k=u+1,\ldots,l. \tag{2.11}$$

It is well known (e.g. [1, II, 1.4] or [5, VII, Sec. 25]) that there exists an analytic matrix $Q(z)$ in the neighborhood of $\zeta - U_\zeta$ such that $Q^{-1}(z)$ is also analytic for $z\in U_\zeta$ and

$$Q^{-1}(z)A(z)Q(z)=B_1(z)\oplus B_2(z), \tag{2.12}$$

where $B_1(z)$ is $m_1\times m_1$ and $B_2(z)$ is $m_2\times m_2$, with

$$m_1=\sum_{i=1}^{u} n_i, \qquad m_2=\sum_{i=u+1}^{l} n_i. \tag{2.13}$$

Moreover the eigenvalues of $B_1(z)$ and $B_2(z)$ are $\mu_1(z),\ldots,\mu_u(z)$ and $\mu_{u+1}(z),\ldots,\mu_l(z)$ with the multiplicities $n_1,\ldots,n_u$ and $n_{u+1},\ldots,n_l$ respectively in U_ζ. Since $A(z)$ is diagonable for each z, $B_1(z)$ and $B_2(z)$ are diagonable in U_ζ. So

$$\prod_{k=u+1}^{l}\left[B_2(z)-\mu_k(z)I\right]=0,$$

which implies

$$\prod_{k=2}^{l}\frac{A(z)-\mu_k(z)I}{\mu_1(z)-\mu_k(z)}$$
$$=Q(z)\left\{\prod_{k=2}^{l}\frac{B_1(z)-\mu_k(z)I}{\mu_1(z)-\mu_k(z)}\oplus 0\right\}Q^{-1}(z). \tag{2.14}$$

The assumption (2.11) and the diagonability of $B_1(z)$ imply

$$B_1(z)=\eta I+\sum_{k=1}^{\infty}(z-\zeta)^k B_k^{(1)}. \tag{2.15}$$

According to Kato [1, II, 2.3] each $\mu_j(z)$ has the Puiseux expansion

$$\mu_j(z)=\eta+\eta_j(z-\zeta)+\sum_{k=s+1}^{\infty}\mu_{jk}(z-\zeta)^{k/s}, \qquad j=1,\dots,u. \tag{2.16}$$

As $\mu_1(z)$ and $\mu_j(z)$ do not touch at ζ, we deduce

$$\eta_1\neq\eta_j, \qquad j=2,\dots,u. \tag{2.17}$$

So

$$\frac{B_1(z)-\mu_j(z)I}{\mu_1(z)-\mu_j(z)}=\frac{\sum_{k=1}^{\infty}(z-\zeta)^{k-1}B_k^{(1)}-\sum_{k=s+1}^{\infty}\mu_{jk}(z-\zeta)^{(k-s)/s}I}{\eta_1-\eta_j+\sum_{k=s+1}^{\infty}(\mu_{1k}-\mu_{jk})(z-\zeta)^{(k-s)/s}}$$

$$j=2,\dots,u. \tag{2.18}$$

Also $[\mu_1(z)-\mu_j(z)]^{-1}$ are analytic in $(z-\zeta)^{1/s}$ in U_ζ for $j>u$. Combine (2.14) and (2.18) to deduce that $P_1(z)$ is analytic in $(z-\zeta)^{1/s}$ in U_ζ. Assume that $\zeta=\infty$, let $\tilde{A}(y)=y^mA(1/y)$, and since A_m is diagonable, transform the point $\zeta=\infty$ to $\zeta=0$, so that we can apply the above analysis to deduce that $P_1(z)$ is analytic in $z^{1/s}$ in the neighborhood of ∞. This in particular implies that $P_1(z)$ is bounded on the Riemann sphere. Put $P_1(z)=(p_{ij}^{(1)}(z))_1^n$. Let $\max|p_{ij}^{(1)}(z)|=|p_{ij}^{(1)}(\zeta_{ij})|$ (ζ_{ij} may be ∞). As $p_{ij}^{(1)}(\zeta_{ij}+w^s)$ is analytic in w, the maximum principle implies that $p_{ij}^{(1)}(\zeta_{ij}+w^s)$ is constant in the neighborhood of the origin (e.g., [1, II, 2.5]). Now the analytic continuation principle yields that $p_{ij}^{(1)}(z)$ is constant. Thus $P_1=P_1(z)$ is a constant matrix. The same argument implies that the projections $P_2,\dots,P_l$ on the eigenvalues $\mu_2(z),\dots,\mu_1 z)$ are constant. To this end let

$$P_i\mathbb{C}=\operatorname{Span}\{y_1^i,\dots,y_{n_i}^i\}, \qquad i=1,\dots,l,$$

$$P=\left(y_1^1,\dots,y_{n_1}^1,\dots,y_1^l,\dots,y_{n_l}^l\right)\in M_n(\mathbb{C}). \tag{2.19}$$

Take a generic point ζ such that

$$\mu_i(\zeta)\neq\mu_j(\zeta) \qquad \text{for} \quad i\neq j, \quad i,j=1,\dots,l. \tag{2.20}$$

So P_i is the projection on $\mu_i(z)$, $z \in U_\zeta$. Thus in this neighborhood

$$P^{-1}A(z)P = \operatorname{diag}(\lambda_1(z), \dots, \lambda_n(z)).$$

As the off-diagonal entries of $P^{-1}A(z)P$ vanish in U_ζ, they must vanish identically, since $P^{-1}A(z)P$ is a polynomial matrix. This establishes (2.9). The proof of the theorem is concluded. ■

The example

$$A(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.21}$$

shows that the first possibility mentioned in Theorem 2.8 may actually happen.

COROLLARY 2.22. *Consider the polynomial* (1.4) *where* $A_0, \dots, A_m$ *are* $n \times n$ *diagonable matrices such that*

$$A_p A_q \neq A_q A_p$$

for some $0 \leqslant p < q \leqslant m$. *Then either there exists* z_0 *(possible infinite) such that two distinct eigenvalues* $\lambda_p(z)$ *and* $\lambda_q(z)$ *touch at this point, or there exists a finite point* z_0 *such that* $A(z_0)$ *is not diagonable.*

Finally, it is left to point out that if $m = 1$, i.e. the pencil $A(z)$ is linear, then the first possibility mentioned in Theorem 2.8 is ruled out. This follows from the fact [4] that in this case the assumption that $A(z)$ is diagonable for any finite z implies that the eigenvalues of $A(z)$ are linear in z. [Following Kato, one notices that the derivative of the multivalued function $\lambda_j(z)$ is bounded on the Riemann sphere and thus $\lambda_j'(z) = \eta_j$, $j = 1, \dots, l$.] Therefore if $\lambda_p(z)$ and $\lambda_q(z)$ touch at z_0, then $\lambda_p(z) \equiv \lambda_q(z)$.

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